On the Ladyzhenskaya-Smagorinsky turbulence model. The regularity problem

H. Beirão da Veiga
Dipartimento di Matematica Applicata "U.Dini"
University of Pisa, Italy
bveiga@dma.unipi.it

Abstract

In the forthcoming paper [2] we consider the difficult question of regularity up the boundary of solutions to boundary value and initial boundary value problems for the modified Navier-Stokes equations. By regularity up to the boundary, we mean the existence of the second derivatives of the velocity and the first derivatives of the pressure which are summable up to the boundary with some exponent.

In [1] we give a solution to this problem for half-space, in the stationary case. The scheme proposed by us in [1] is as follows. At first, the steady state problem with no convection term is considered. It is a kind of the nonlinear Stokes problem. For values of the exponent $p \in [2,3]$, characterizing the growth of the dissipative potential, the starting regularity is proved. More precisely, it is shown that the second derivatives of the velocity field are summable with power $p' = p/(p-1)$. Then this exponent is improved up to some exponent $l > p'$ if $p < 3$, with help of bootstrap arguments. For $p = 3$ one has $l = p'$, and bootstrap is not needed. For $p = 2$, the result is the same as in the case of the classical Stokes problem with the right hand side from L_2. All the results remain to be true in the presence of the convective term. For $p > 2$, its role is not so crucial.

The main aim of [2] is to extend the above scheme to the case of curvilinear boundaries. This is a quite difficult technical problem. Especially, it is complicated for nonlinear equations containing viscosity depending on the module of $\mathcal{D}$, the symmetric part of the gradient. Actually, the latter circumstance makes the given problem so difficult even in the case of flat boundaries. For it, the known scheme developed for the case in which coefficients in the equations depends on the module of the gradient (Lions model), does not work.

The proof of our results is done via a very careful analysis up to boundary, and a suitable application of a modified difference quotient method (and this is the

relevant novelty of the paper) overcoming the simultaneous appearance of three difficulties: boundary regularity (that is, how to recover the vertical derivatives of $\mathcal{D}$ from the tangential ones), the divergence constraint to be met at each choice of the test functions, and the fact that the system actually depends on the symmetric part of the gradient, rather than on the gradient itself. This leads to the introduction of a certain number of interesting new tricks. The results are anyway proved by first arguing locally, via a suitable flattening of the boundary, and then by a covering argument to recover the final global estimate. A main scientific importance of our work is that it shows that all the results being valid for flat boundaries remain to be valid for general smooth domains. In a paper in preparation we prove corresponding results for the shear thinning case, $p < 2$.

Keywords: Navier-Stokes equations, Shear thickening viscosity, Regularity up to the boundary.

References

[1] Beirão da Veiga, H. On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip and non-slip boundary conditions, *Comm. Pure Appl. Math.*, **58** (2005), 552-577.

[2] Beirão da Veiga, H. On the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations in smooth domains. The regularity problem, *J. Euro. Math. Soc.*, submitted.