A particle method for the steady-state Navier-Stokes equations

Werner Varnhorn
Faculty of Mathematics
University of Kassel, Germany
varnhorn@mathematik.uni-kassel.de

Abstract

For the description of fluid flow there are in principle two approaches, the Eulerian approach and the Lagrangian approach. The first one describes the flow by its velocity $v = (v_1(x), v_2(x), v_3(x)) = v(x)$ in every point $x = (x_1, x_2, x_3)$ of the domain G containing the fluid. The second one uses the trajectory $x = (x_1(t), x_2(t), x_3(t)) = x(t) = X(t, x_0)$ of a single particle of fluid, which at initial time $t = 0$ is located at some point $x_0 \in G$. The second approach is of great importance for the numerical analysis and computation of fluid flow, while the first one has also often been used in connection with theoretical questions.

We consider the stationary motion of a viscous incompressible fluid in a bounded domain $G \subset \mathbf{R}^3$ with a sufficiently smooth boundary S. Because for steady flow the streamlines and the trajectories of the fluid particles coincide, both approaches mentioned above are correlated by the autonomous system of characteristic ordinary differential equations

$$x'(t) = v(x(t)), \quad x(0) = x_0 \in G, \tag{1}$$

which is an initial value problem for $t \longrightarrow x(t) = X(t, x_0)$, if the velocity field $x \longrightarrow v(x)$ is known in G. To determine the velocity, in the present case we have to solve the steady-state nonlinear equations

$$-\nu\Delta v + v\cdot\nabla v + \nabla p = F \quad \text{in} \quad G, \qquad \operatorname{div} v = 0 \quad \text{in} \quad G, \qquad v = 0 \quad \text{on} \quad S \tag{2}$$

of Navier-Stokes. Here $x \longrightarrow p(x)$ is an unknown kinematic pressure function. The constant $\nu > 0$ (kinematic viscosity) and the external force density F are given data. The incompressibility of the fluid is expressed by $\operatorname{div} v = 0$, and on the boundary S we require the no-slip condition $v = 0$. It is the aim of the present lecture to develop a new approximation method for (2) by coupling both the Lagrangian and the Eulerian approach. The method avoids fixpoint considerations and leads to a sequence of approximate systems, whose solution is unique and has a high degree of regularity, important at least for numerical purposes. Moreover, we show that our method allows the construction of weak solutions of the Navier-Stokes equations: The sequence of approximate solutions has at least one accumulation point satisfying (2) in a weak sense.

Keywords: Navier-Stokes equations, lagrangian approximation, weak solution.

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