A stability theorem of Navier-Stokes flow past a rotating rigid body

Yoshihiro Shibata
Department of Mathematical Sciences
School of Science and Engineering
Waseda University, Japan
yshibata@waseda.jp

Abstract

In this talk, I would like to consider the Navier-Stokes equations describing a viscous incompressible flow past a rotating rigid body in the 3-dimensional Euclidean space $\mathbb{R}^3$ with axis of rotation $\omega = a\beta_3$ ($\beta_3 = (0,0,1)$) and with velocity $u_\infty = k\beta_3$ where a and k are both non-zero positive constants. After transformations of coordinate and unknown functions, we have the following equations:

$$\begin{aligned} u_{tt} + u\cdot\nabla u - \Delta u + k\partial_3 u - (\omega\times x)\cdot\nabla u + \omega\times u + \nabla p = f \\ \operatorname{div} u = 0 \end{aligned} \tag{1}$$

in a fixed exterior domain Ω with $C^{1,1}$ boundary Γ, subject to the initial and boundary conditions:

$$\begin{aligned} &u(x,t) = \omega\times x - u_\infty && \text{on } \Gamma \\ &u(x,0) = u_0(x) && \text{in } \Omega \\ &\lim_{|x|\to\infty} u(x,t) = 0 \end{aligned}$$

Here, $u = (u_1, u_2, u_3)$ and p denote the unknown velocity and pressure of the fluid, respectively; f is a given external force assumed to be a function depending only on x, that is $f = f(x)$.

The stationary problem corresponding to equation (1) is:

$$\begin{aligned} &v\cdot\nabla v - \Delta v + k\partial_3 v - (\omega\times x)\cdot\nabla v + \omega\times v + \nabla\theta = f && \text{in } \Omega \\ &\operatorname{div} v = 0 && \text{in } \Omega \\ &v(x) = \omega\times x - u_\infty && \text{on } \Gamma \\ &\lim_{|x|\to\infty} v(x) = 0 \end{aligned} \tag{2}$$

Galdi and Silvestre [1],[2] proved a unique existence theorem of equation (2) for rather small $k > 0$ and f in a certain norm. Moreover, they investigated a wake behaind the body mathematically. Namely, their solutions have the following estimate:

$$
\begin{aligned}
|v(x)| &\le M(1+|x|)^{-1}(1+k(|x|+x_3))^{-1} \\
|\nabla(x)| &\le M(1+|x|)^{-3/2}(1+k(|x|+x_3))^{-3/2}
\end{aligned} \tag{3}
$$

where M is a constant independent of k and a provided that $k \in [0, B_1]$ and $a \in [0, B_2]$.

In this talk, I am interested in some stability result of the Galdi - Silvestre stationary solutions with respect to small initial disturbance, in other words the attainable problem. To explain what I want to show, I set $u(x,t) = v(x)+w(x,t)$ and $p(x,t) = \theta(x)+\pi(x,t)$, where u and p are unknow functions of the original problem (1). Then, w and π should satisfy the following equations:

$$
\begin{aligned}
&w_{tt} + v(x)\cdot\nabla w + w\cdot\nabla v(x) + w\cdot\nabla w \\
&\quad -\Delta w + k\partial_3 w - (\omega\times x)\cdot\nabla w + \omega\times w + \nabla\pi = 0 && \text{in } \Omega\times(0,\infty) \\
&\operatorname{div} w = 0 && \text{in } \Omega\times(0,\infty) \\
&w(x,t) = 0 && \text{on } \Gamma\times(0,\infty) \\
&w(x,0) = w_0(x) && \text{in } \Omega \\
&\lim_{|x|\to\infty} w(x,t) = 0
\end{aligned} \tag{4}
$$

I would like to report that equation (3) admits a unique solution $w \in C^0([0,\infty), PL_3(\Omega)) \cap C^0((0,\infty), W_3^1(\Omega)\cap L_\infty(\Omega))$ provided that $w_0 \in PL_3(\Omega)$, $\|w_0\|_{L_3}$ is small enough and $v(x)$ and $\nabla v(x)$ are also small enough in a certain norm, where P denotes the Helmholtz-Weyl-Leray projection. Moreover, w has the following asymptotic behaviour as t goes to infinity:

$$
\begin{aligned}
\|w(\cdot,t\|_{L_q} &\le Ct^{-\left(\frac{1}{2}-\frac{3}{2q}\right)} \quad (3\le q\le\infty) \\
\|\nabla w(\cdot,t)\|_{L_3} &\le Ct^{-\frac{1}{2}}
\end{aligned}
$$

When $a = 0$ (no rotation and only past a rigid body case), in [3] I proved the same result, so that this is an extension to the case when $a \neq 0$.

To prove such stability result, the point is to show so called L_p-L_q decay estimate of the Oseen semitroup with rotating effect. To state this more precisely, let me consider the corresponding linearized equation to (3):

$$
z_t + \mathcal{L}_{k,a}z = 0 \quad (t>0), \quad z|_{t=0} = z_0 \tag{5}
$$

where $\mathcal{L}_{k,a}$ is defined as follows:

$$
\begin{aligned}
\mathcal{D}_q(\mathcal{L}_{k,a}) &= \{f\in JL_q(\Omega)\cap W_q^2(\Omega) \mid (\omega\times x)\cdot\nabla f\in L_q(\Omega), \quad f|_\Gamma = 0\} \\
\mathcal{L}_{k,a}z &= P(-\Delta z + k\partial_3 z - (\omega\times x)\cdot\nabla z + \omega\times z) \quad (z\in\mathcal{D}_q(\mathcal{L}_{k,a}))
\end{aligned}
$$

Then, we have the following theorem.

Theorem 1 (1) *Let* $1 < q < \infty$. *Then,* $\mathcal{L}_{k,a}$ *generates a* C^0 *semigroup* $\{T(t)\}_{t\geq 0}$ *on* $PL_q(\Omega)$.
(2) $\{T(t)\}_{t\geq 0}$ *has the following estimates:*

$$\|T(t)f\|_{L_r} \leq C_{q,r} t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L_q} \qquad t > 0$$

($1 \leq q \leq r \leq \infty$, $r \neq 1$, $q \neq \infty$*)*

$$\|\nabla T(t)f\|_{L_r} \leq C_{q,r} t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L_q} \quad t > 0$$

($1 \leq q \leq r \leq 3$, $r \neq 1$*).*

Applying the argument in [3] based on the argument due to Kato [4] and using Theorem 1 and estimate (5), we can show the stability theorem.

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