An existence theorem for the steady Navier–Stokes equations

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Abstract

In the annulus $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < R\}$ we consider the Navier–Stokes problem

$$\begin{aligned} \Delta u - \lambda(\nabla u)u &= \nabla p \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= a \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $u : \Omega \to \mathbb{R}^2$, $p : \Omega \to \mathbb{R}$ are the (unknown) kinetic and pressure fields respectively, λ the Reynolds number and $a \in C^\infty(\partial\Omega)$ the boundary datum satisfying the necessary condition $\int_{\partial\Omega} a \cdot n = 0$, where n denotes the outward unit normal to $\partial\Omega$. This problem has been the object of several researches since the appearance in 1933 of a famous paper of J. Leray, where it is proved that system (1) has a regular solution, provided $\Phi = \int_{|x|=1} a \cdot n = 0$. This assumption was relaxed by G.P. Galdi in 1992 and W. Borchers and K. Pileckas in 1994, who showed that Leray's result still holds provided $\lambda|\Phi|$ is sufficiently small.

The first result on existence of a solution to system (1) without any hypothesis on Φ was proved by C. J. Amick in 1984 under suitable assumptions of symmetry on Ω and $a = (a_1, a_2)$. Precisely, he proved that, if Ω is symmetric with respect to the x_1–axis and a_1 is a pair function with respect to x_2 and a_2 is an odd function with respect to x_2, then system (1) has a solution for every value of Φ. To the best of our knowledge this is, concisely, the state of art of the Navier–Stokes problem in the annulus. As a consequence, the fundamental question concerning it, *i.e*, whether system (1) is solvable for every value of Φ is still open.

Starting from some results and suggestions contained in the just quoted paper of C.J. Amick, we give a contribution to the above problem by proving that (1) is solvable for every λ provided $\Phi \geq 0$ (outflow condition).

Keywords: two dimensional bounded domains, Stokes system, stationary Navier–Stokes equations, boundary–value problem.