Solvability of the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$ for some class of initial data

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Abstract

Existence of a unique regular solution of the Cauchy problem in $\mathbb{R}^3$ for the Navier-Stokes equations

$$\partial_t v(x,t) - \Delta v(x,t) + (v(x,t)\cdot\nabla)v(x,t) + \nabla p(x,t) = 0,$$

$$\mathrm{div} v(x,t) = 0,$$

$$v(x,0) = v_0(x),$$

where

$$\mathrm{div} v_0(x) = 0.$$

is proved for the class of initial data v_0 that have some finite weighted norm and suppv_0 belongs to $\mathbb{R}^3 \setminus B_{R_0}$, where B_{R_0} is a ball with sufficiently large radius R_0. The proof follows from appropriate estimates in weighted Sobolev spaces. The results were obtained jointly with W. M. Zajączkowski.

Let

$$A^2_{0,\beta} = \int\limits_{\mathbb{R}^3} (1+|x|^2)^{\beta+1}\left[\frac{|v_0(x)|^2}{1+|x|^2} + |\nabla v_0(x)|^2\right]dx + \Bigg(\int\limits_{\mathbb{R}^3} |v_0(x)|dx\Bigg)^2,$$

if $\beta \in [1, 3/2)$, and

$$A^2_{0,\beta} = \int\limits_{\mathbb{R}^3} (1+|x|^2)^{\beta+1}\left[\frac{|v_0(x)|^2}{1+|x|^2} + |\nabla v_0(x)|^2\right]dx + \Bigg(\int\limits_{\mathbb{R}^3} (1+|x|^2)^{1/2}|v_0(x)|dx\Bigg)^2,$$

$\beta \in [3/2, 5/2)$. The main result read as follows.

Theorem. *Let* $\mathrm{div} v_0(x) = 0$ *and* $\mathrm{supp} v_0 \subset \{x \in \mathbb{R}^3 : |x| > R_0\}$. *Assume that for* v_0 *the norm* $A_{0,\beta}$ *is finite. There exists an absolute constant* c_* *such that if*

$$A_{0,\beta}^2 \le c_* R_0^{2\beta+3/2}(1+T)^{-1}, \tag{1}$$

then the Cauchy problem for the Navier-Stokes equations admits a unique regular solution $(v(x,t), p(x,t))$ *on the interval* $[0,T]$.

Consider the following example. Let $\varphi(x) = (|x|^{-\gamma+1}, |x|^{-\gamma+1}, |x|^{-\gamma+1})$ and let $\zeta(s)$ be a smooth cut-off function with $\zeta(s) = 1$ for $2R_0 < s < 3R_0$ and $\zeta(s) = 0$ for $s \le R_0$ and $s \ge 4R_0$. Define

$$v_0(x) = \mathrm{curl}(\zeta(|x|)\varphi(x)). \tag{2}$$

Then $\mathrm{div} v_0 = 0$, $\mathrm{supp} v_0 \subset \{x \in \mathbb{R}^3 : R_0 \le |x| \le 4R_0\}$ and, obviously, the norm $A_{0,\beta}$ is finite. If

$$\gamma \in (3/4, 1), \qquad \beta \in (9/4 - \gamma, 3/2), \tag{3}$$

then

$$\|v_0\|_{L_2(\mathbb{R}^3)} \to \infty, \quad \|v_0\|_{L_3(\mathbb{R}^3)} \to \infty \quad \text{as} \quad R_0 \to \infty$$

and it is easy to compute the estimate for T

$$1 + T \le \frac{C R_0^{2\beta+3/2}}{R_0^{2\beta+3-2\gamma} + R_0^{6-2\gamma}} \to \infty \quad \text{as} \quad R_0 \to \infty.$$

Thus, under condition (3), the L_3 - norm of the initial data given by the formula (2) grows as the support of v_0 moves to infinity and, simultaneously, the solvability interval $[0,T]$ becomes arbitrary large as $R_0 \to \infty$.

The condition (1) may be interpreted as some kind of smallness assumptions for the weighted norm $A_{0,\beta}$. However, this condition do not requirer $L_3(\mathbb{R}^3)$–norm of the initial datum to be small. Note that the condition (1) is not scaling invariant.

Keywords: Navier-Stokes equations, regularity problem, weighted Sobolev spaces.