Harmonic vector fields in 3D bounded domains and its application to the stationary Navier-Stokes equations

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Abstract

The first aim of this talk is to show some decomposition theorem on the vector fields in $L^r(\Omega)$ for $1 < r < \infty$, where Ω is a bounded domain in $\mathbf{R}^3$ with C^∞-boundary $\partial\Omega$. Let us define two spaces $V_{har}(\Omega)$ and $X^r_\sigma(\Omega)$ by

$$V_{har}(\Omega) \equiv \{h \in C^\infty(\overline{\Omega}); \mathrm{div}\ h = 0, \quad \mathrm{rot}\ h = 0 \quad \text{in } \Omega, \quad h \times \nu = 0 \quad \text{on } \partial\Omega\},$$
$$X^r_\sigma(\Omega) \equiv \{w \in W^{1,r}(\Omega); \mathrm{div}\ w = 0 \quad \text{in } \Omega, \quad w \cdot \nu = 0 \quad \text{on } \partial\Omega\},$$

where ν is the unit outer normal to $\partial\Omega$. Then it holds that

$$L^r(\Omega) = V_{har}(\Omega) \oplus \mathrm{rot}\ X^r_\sigma(\Omega) \oplus \nabla W^{1,r}_0(\Omega), \quad 1 < r < \infty \quad \text{(direct sum)}.$$

As an application of our decomposition theorem, we consider the *inhomogeneous* boundary value problem of the stationary Navier-Stokes equations in Ω when $\partial\Omega$ consists of $N+1$-disjoint C^∞-surfaces $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$, where $\{\Gamma_j\}_{j=1}^N$ lie in Γ_0;

$$\text{(N-S)} \qquad -\mu\Delta u + u \cdot \nabla u + \nabla p = 0, \quad \mathrm{div}\ u = 0 \quad \text{in } \Omega\ , \quad u = \beta \quad \text{on } \partial\Omega.$$

Here $\mu > 0$ is the viscosity constant and $\beta \in H^{1/2}(\partial\Omega)$ is the given boundary data on $\partial\Omega = \cup_{j=0}^N \Gamma_j$.

We take N-harmonic functions $q_1, \cdots, q_N$ in Ω so that $q_j|_{\Gamma_0} = 0$ and $q_j|_{\Gamma_k} = \delta_{jk}$ for $j, k = 1, \cdots, N$. Define $\psi_j \equiv \nabla q_j$, $j = 1, \cdots, N$. It is shown that $\{\psi_1, \cdots, \psi_N\}$ is a basis of $V_{har}(\Omega)$. Taking the Gramm matrix $\{\alpha_{jk}\}_{1 \leqq j,k \leqq N}$ defined by $\{\psi_1, \cdots, \psi_N\}$, we see that $\varphi_j \equiv \sum_{k=1}^N \alpha_{jk}\psi_k$, $j = 1, \cdots, N$ is an orthogonal basis of $V_{har}(\Omega)$ in the sense of $L^2(\Omega)$. Then we have

Theorem. *Let $\beta \in H^{1/2}(\partial\Omega)$ satisfy the general flux condition* $\displaystyle\sum_{j=0}^N \int_{\Gamma_j} \beta \cdot \nu dS = 0.$

If

$$\left\| \sum_{j,k=1}^N \alpha_{jk} \left(\int_{\Gamma_k} \beta \cdot \nu dS \right) \varphi_j \right\|_{L^3(\Omega)} < \mu C_s^{-1},$$

then there exists at least one weak solution $u \in H^1(\Omega)$ *of (N-S). Here* $C_s = 3^{-\frac{1}{2}} 2^{\frac{2}{3}} \pi^{-\frac{2}{3}}$ *is the best constant of the Sobolev embedding* $H^1_0(\Omega) \subset L^6(\Omega)$.

Keywords: harmonic vector fields, general flux condition, Leray's inequality